"THE RESULTS OF DEVELOPING AND TESTING AN EXPERIMENTAL APPARATUS FOR THE SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS", (Note: Presented 23 Dec 1946 at Seminar on Precise Mechanics and Calculating Techniques of Dept of Precise Mech of the Institute of Mech Sci.)

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(Submitted by Acad. N. G. Bruyevich).

A great many technical problems lead to the necessity of solving a system of differential equations (DE) with constant coefficients. The most important of them are the problems of calculating dynamical systems; for example, the calculation of dynamical stability and of small oscillations in an airplane and the investigation of stability of a different kind in regulators and tracking systems, etc.

In most practical cases, the various authors are limited more or less to detailed discussions of stability or they do not consider a dynamical process at all, or give one or two, in the best cases, examples. This is due to the great complexity and difficulty of calculating a dynamical process.

The creation of an apparatus that would permit one to obtain a series of curves describing the investigated dynamical process is very necessary.

The author, on the basis of an idea of L. I. Gutenmakher, developed an experimental apparatus which permits one very quickly, in 5 to 10 minutes, to set up the system of equations under study and afterwards in several seconds to obtain the family of curves for various values of the coefficients.

L. I. Gutenmakher proposed with the aid of amplifiers to create a physical model-analogy of a dynamical system.

During the development of the universal apparatus, which permits one quickly to set up the models of dynamical systems, it seemed expedient to give up accurately copying the scheme of construction of a dynamical system, as was earlier done by a certain author by way of substitution of the friction element by a resistor, mass element by a self-induction coil and elastic element by a capacitor etc.

Instead of "copying", there was constructed a universal apparatus which permits one to set up models whose processes are described by a system of differential equations of only the first order.

As is known, a system of DE (or one equation) of high order can always be transformed into a system of DE of the first order.

The theoretical scheme or principle of the apparatus is represented in Figure 1 (Note: All figures are in the annex). The circuit scheme consists of groups of amplifiers connected into a single system with the help of ohmic resistance and capacitance.

The circuit is constructed in such a manner that the output of each amplifier can be joined with the input of any other amplifier or through a resistor or capacitor.

As a result of such a hook-up, as was shown by L. I. Gutenmakher (1, 2), one obtains an electrical model whose processes are described by a system of ordinary linear DE of the first order with constant coefficients.

In the most general case the system can possess the following form:

$$A_{11}U_1 + b_{11}\frac{du_1}{dt} + A_{12}U_2 + b_{12}\frac{du_2}{dt} + \cdots + A_{nn}U_n + b_{nn}\frac{du_n}{dt} = F_1(t)$$

$$A_{n1}U_1 + b_{n1}\frac{du_1}{dt} + A_{n2}U_2 + b_{n2}\frac{du_2}{dt} + \cdots + a_{nn}U_N + b_{nn}\frac{du_n}{dt} = F_n(t) \qquad (1)$$

The variation of the coefficients of these equations is effected by switching in resistances (for a_{ik}) or capacitances (for b_{ik}) on the voltage dividers which are connected to the output of the amplifiers (2).

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The initial conditions are assigned by charging all capacitors of the circuit through special capacitors which are joined to the input of the amplifiers.

The method of assigning the right part $F_1(t)$, as used by the author, is similar to the method employed in V. Bush's integrator (3). The essence of our method is easily explained by an example of a system of nonhomogeneous equations:

$$(a_{11}+b_{11}p)\,x + (A_{12}+b_{12}p)\,y = f_1(t)$$
$$(A_{21}+b_{21}p)\,x + (A_{22}+b_{22}p)\,y = f_2(t) \qquad \left(p \equiv \frac{d}{dt}\right) \tag{2}$$

This system can be replaced by a system of four equations with four unknowns x, y, u, v:

$$(A_{11}+b_{11}p)\,x + (A_{12}+b_{12}p)\,y - \alpha u = 0$$
$$(A_{21}+b_{21}p)\,x + (A_{22}+b_{22}p)\,y - \beta v = 0. \tag{3}$$

$$px - g_1\,(t, u, v) = 0$$
$$pv - g_2\,(t, u, v) = 0. \qquad [Sic.] \tag{4}$$

In this system the first two equations are linear and homogeneous, but the two latter equations are merely "auxiliary" unknowns in u and v. The functions g_1 and g_2 are so selected that the solution of the system consisting of the two latter equations in (4) are satisfied by the functions u and v, which are proportional to the right parts of the given equations. The coefficients α, β are chosen in dependence upon the desired scale of the solution. In each case it is necessary that:

$$\alpha u(t) = f_1(t) \text{ and } \beta v(t) = f_2(t). \tag{4^1}$$

If the functions $f_1(t)$ and $f_2(t)$ represent the sums of derivatives of exponential functions and polynomials in t, that is, if they have the form:

$$\sum_{k=1}^{s_j} e^{\alpha_{k_j} t} \sum_{l=1}^{q_k} a_{jl} t^{l} \quad (j=1,2) \qquad (5)$$

(Note: The coefficients A_{kj} can be even complex which makes it possible to obtain functions of the form $e^{nt} \cdot \sin (mt + \phi)$.);then the linear equations with constant coefficients can be related as equations determining the functions u and v. (Note: The additional system of equations is easily obtained by the use of the method of Laplace transforms. The application of this method will be discussed in another paper.)

In this manner, one can, in order to solve a system of nonhomogeneous equations, utilize the remaining free amplifiers for assigning the right parts. Below we shall give examples showing the application of this method to concrete problems.

In order give the right side in the form of an arbitrary function of time, we have not excluded the possibility of employing a special rotating commutator or "figure" rheostat.

Description of the Construction of the Experimental Apparatus

The apparatus (Figures 2, 3) consists of the following units:

1) A group of amplifiers (6 pieces).
2) Matrix of conductances (basic commutation panel).
3) Panel of given initial conditions.
4) Device for assigning initial conditions and "damping".

The entire apparatus represents a vertical structure about two meters high, and 1 meter wide and 35 centimeters "thick" (when covered in a case). The amplifiers and conductance matrix are disposed on a vertical panel of laminated insulation. The panel for assigning initial conditions are placed horizontally below the apparatus and possess a width of about 20 cm. Still lower is placed the device, for "damping" and for assigning initial conditions, which consists of two relays.

The role of the amplifiers, to be discussed below, is such that all the coefficients of the equations can be assigned independently of one another. In order to obtain positive or negative coefficients the amplifier has two dividers.

The conductance matrix (or basic commutation panel) is intended to effect the hook-up of each amplifier with any other through a resistor or capacitor.

The panel of the conductance matrix is divided into 36 rectangular cells or squares (6 x 6). In each of the cells of one vertical column are located the outputs of a divider.

Along the horizontal rows of the cells are disposed copper busbars i connected to the inputs of the amplifiers (the first busbar above to the first amplifier, the second bar to the second amplifier, etc.)

To each busbar are connected, by several terminals, coupling conductances which are made in the form of carbon resistors k (type "CC") of about 10^5 ohms and in the form of mica capacitors l (type B) of about 10^4 micromicrofarads.

The second terminals of the coupling conductances (m and n) can be joined to the outputs from a divider with the help of short flexible cables.

Besides coupling conductances, to the cells i are also connected capacitors for the assignment of initial conditions, and resistors of constant feedback, necessary for compensating for the active conductance at the input.

On the panel for the assignment of initial conditions there are six pairs of potentiometers, 200 ohms. To the sliding runners of one of the two potentiometers of each pair (depending upon sign) one can, with the help of flexible cables, join the capacitors for assigning the initial conditions previously mentioned. All pairs of potentiometers are joined in parallel with the help of copper busbars, to which at the initial moment is connected a battery of 30-40 volts at all times of the operating cycle. Due to the hookup of the battery, all capacitors of the circuit obtain charges depending upon the values of voltage established in the potentiometers.

The necessary voltages on the potentiometers e_{io} are determined according to following simple formula:

$$e'_{io} = b_{i1}u_{10} + b_{i2}u_{20} + \cdots + b_{in}u_{no} = \sum_{k=1}^{n} b_{ik}u_{ko} \quad (i = 1, 2, \cdots n)$$

where b_{ik} are coefficients in the derivatives and u_{ko} are initial values of the functions.

The connecting in of the battery is conducted from two relays. The periodic contact of the relays is attained by means of a cam mechanism fastened either on the shaft of a rotating oscillograph mirror or on the shaft of a small Warren motor making one revolution a second. The time of closing of the contacts is about 0.1 second. The relays serve at the same time also for damping the system (that is, for discharging all capacitors). The damping is effected by the closing of the inputs of the amplifiers for a time of about 0.9 seconds.

The Amplifier Circuit Scheme

As shown by theoretical and experimental investigations, the amplifier has to face very strict requirements, particularly in connection with the constancy of the amplification factor. We shall enumerate several of these.

1. It is necessary that the amplification factor differ from the rated quantity not more than by 0.3% for a phase shift between the output and input not greater 0.2° (on a sufficiently large range of frequenàies we have $f_{max}/f_{min} \geqslant 100$). (Note: If one of these requirements is not fulfilled, then, for example, in the case of the solution of a system of two first-order equations, there is obtained, instead of a nondamping oscillatory process, either a damping or a non-increasing process with a velocity of amplitude variation greater than 2% over the period.)

2. The resistance of a divider must be at least 150 times less than one resistance of the coupling.

The author worked out a scheme of an amplifier (Figure 4) which gave good results. Theoretical calculations showed that the operating range of frequencies, in which the angle of phase shift is less than 0.2°, ranges from 1.6 to 320 cycles.

Experimental investigations confirmed these calculations. Moreover, experiments showed that the amplification factor is very stable.

Thus for example:

a) For a variation in the magnitude of the assumed voltage from 0 to the limiting amplitude 38 volts the nonlinear distortion does not exceed 0.2%.

b) The stability of the amplification factor in the limits ±0.3% is reached for oscillations of the feed voltage in the limits ± 6%.

c) The substitution switching-around of tubes varies the amplification factor in the limits ± 2%. This variation can easily be compensated for by adjustment of the amplification factor with the help of a special shaft.

The author worked out a special method for the quick control of the amplification factor. It is sufficient to set in the apparatus the equation $\frac{du_i}{dt}$ equal to zero and to solve it for the initial conditions $u_{io} = U$, where U is a certain arbitrary voltage.

If the amplification factor differs from the rated quantity by $\mp\varepsilon$ (%), then in this case actually the following equation will be solved

$$\frac{du_i}{dt} \pm \varepsilon(\%)\cdot 10^{-2}\, u_i = 0 \qquad (6^1)$$

and instead of the solution of the given equation u_i = constant we shall obtain:

$$u_i = U\exp\left(\mp\frac{\varepsilon(\%)}{100}\cdot\frac{t}{T}\right) \qquad (7)$$

where T = RC is the coupling's time constant.

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As a result, on the oscillograph there will appear, instead of a line parallel to the abscissa axis, a curve coming from below or above depending upon the sign of epsilon ε.

The sensitivity of the method is very great, which is easily seen from the formula:

$$\Delta(\%) \approx \varepsilon(\%) \cdot \frac{t}{T} \qquad (8)$$

where delta Δ (%) is the variation in voltage at the moment of time t.

For example, for t = 0.1 sec, Δ(%) = 20% and T = 10^{-3} second, we then have:

$$\varepsilon(\%) = \frac{\Delta(\%)T}{t} = \frac{20 \cdot 10^{-3}}{10^{-1}} = 0.2\ \% \qquad (8^1)$$

Moreover, it was found that this method permits one to control the working order of the entire apparatus effectively.

If, for example, one disrupts the contact of any of the coupling resistors, then because of the decrease in conductance at the input of one of the amplifiers the curve of adjustment becomes sharp upwards. If one disrupts the contact of any coupling capacitor, then the curve of adjustment at N = N-rated will remain parallel to the abscissa axis, but its entire ordinate axis will be magnified approximately by 17%, which is also easily shown.

As the theoretical investigation of I. S. Gradshteyn and V. A. Taft, (4, 5) show, in certain cases, thanks to the influence of "parasitic" capacitances in an amplifier, the apparatus can give a false solution.

Ordinarily, a false solution is characterised by the appearance of rapid oscillatory processes which do not disappear even for zero initial conditions.

I. S. Gradshteyn showed a simple method which permits one to get rid of, completely, "parasitic" oscillations; to do this it is sufficient to reduce the matrix of coefficients for arbitrary b_{ik} to trigonometric form; that is, the coefficients b_{ik} must be different from zero, the indices of which being connected by the inequality $i \geq k$ (or $k \geq i$). The reduction to trigonometric form is effected very simply by the summation of the individual equations of the system multiplied by a certain coefficient.

Examples of the Solution of Problems

Very instructive are the problems on the integration of a constant quantity, since in this case the entire error is integrated. Figure 5 shows the results of one ($u_2(t)$), two ($u_3(t)$), three ($u_4(t)$), and four ($u_5(t)$) fold integration of a constant.

The problem of integrating a constant is equivalent to solving a system of equations of the following form:

$$
\begin{aligned}
& b_{11}\frac{du_1}{dt} = 0 \\
& b_{22}\frac{du_2}{dt} - A_{21}U_1 = 0 \qquad \text{for } U_{10} = U \\
& b_{33}\frac{du_3}{dt} - A_{32}U_2 = 0 \\
& b_{44}\frac{du_4}{dt} - A_{43}U_3 = 0 \qquad u_{20} = u_{30} = u_{40} = u_{50} = 0 \\
& b_{55}\frac{Du_5}{dt} - A_{54}U_4 = 0
\end{aligned} \tag{9}
$$

The points drawn on the oscillograms show the true values of the function at a given moment of time.

As is seen from the oscillograms, the error in this case is from 2 to 4%.

In Figure 6 is shown the solution of the following first-order DE:

$$b_{11}\frac{du_1}{dt} + a_{11}u_1 = 0 \text{ for } u_{10} = U \tag{10}$$

The error in the magnitude of the time constant is of the order 1%.

In Figure 7 are shown two cases of the solution of the following two equations of the first order:

$$b_{11}\frac{du_1}{dt} - a_{11}u_1 + a_{12}u_2 = 0$$

$$b_{22}\frac{du_2}{dt} - a_{21}u_1 = 0 \qquad \text{for } u_{10} = U \qquad (11)$$

$$u_{20} = 0.$$

The error in frequency is of the order 2%; but in the magnitude of the time-constants of damping (non-increasing), it is in both cases not greater than 1.5%.

For a non-dampened process, a_{11} must be equal to zero (Figure 8).

The solution is given for arbitrary initial conditions; the error in frequency is of the order 1%. The error in amplitude is also not greater than 1%. The error in the initial phase of oscillations is not greater than 3 degrees.

Figure 9 shows the solution of the following second-order equation with the right part having the form $e(t) = kt$: (12)

$$\frac{d^2u_1}{dt^2} + \omega^2 u_1 = kt \quad \text{for } u_{10} = 0;\ \left(\frac{du_1}{dt}\right)_0 = 0.$$

To solve on the integrator, this equation is transformed into a system of two first-order equations (by replacing du1/dt by a new variable u_2):

$$\frac{du_1}{dt} - u_2 = 0$$

$$\frac{du_2}{dt} + \omega^2 u_1 = kt \quad \text{for } u_{10} = u_{20} = 0. \qquad (13)$$

In order to assign a right part having the form kt, it is necessary to solve a supplementary system of equations. A function of the form kt can be obtained as the result of solving the following DE:

$$\frac{du_3}{dt} - u_4 = 0 \qquad \text{for } u_{30} = 0 \qquad (14)$$

$$\frac{du_4}{dt} = 0 \qquad u_{40} = k.$$

For these conditions $u_3 = kt$.

As a result, in order to solve this problem, the following system of four DE are obtained:

$$\begin{aligned} &\frac{du_1}{dt} - u_2 = 0 & & \\ &\frac{du_2}{dt} + \omega^2 u_1 - u_3 = 0 & &\text{for } u_{10} = u_{20} = u_{30} = 0 \\ &\frac{du_3}{dt} - u_4 = 0 & &u_{40} = k. \end{aligned} \qquad (15)$$

The solution of such a system has the form;

$$u_2(t) = \frac{k}{W^2} \cdot \left(t - \frac{1}{W} \sin Wt\right) \qquad (W \approx \omega). \qquad (16)$$

The error in frequency is of the order 1%. The error in average speed of growth is about 2%.

The variation in the nature of the curve in the upper part is explained by the arrival into the nonlinear region of the amplifier's characteristic.

Figure 10 gives the solution of a second-order equation with the right part having the form e^{-nt} (namely, the upper curve drawn on the oscillogram).

The function e^{-nt} was obtained as a result of solving the equation of the following form:

$$b_{11}\frac{du_1}{dt} + a_{11}u_1 = 0 \text{ for } u_{10} = U \quad \left(\frac{a_{11}}{b_{11}} = n\right) \qquad (17)$$

The system of equations in effect has the following form:

$$\left.\begin{aligned} &b_{11}\frac{du_1}{dt} + a_{11}u_1 = 0 \\ &b_{22}\frac{du_2}{dt} - a_{21}u_1 + a_{23}u_3 = 0 \\ &b_{33}\frac{du_3}{dt} + a_{33}u_3 - a_{32}u_2 = 0 \end{aligned}\right\} \quad \begin{aligned} &\text{for } u_{10} = U \\ &u_{20} = u_{30} = 0 \end{aligned} \qquad (18)$$

The solution of such a system has the following form:

$$u_2(t) = \frac{k}{(n+\alpha)^2 + \beta^2}\left\{e^{nt} - \frac{e^{\alpha t}}{\beta}\left[(n-\alpha)\sin\beta t + \beta\cos\beta t\right]\right\} \qquad (19)$$

The drawn curve is in good agreement with the oscillogram; this is verified by the fact that the error in the magnitude of the time constant and in the frequency does not exceed 2%.

We shall give still another example for the solution of an equation with a right part where the right part has the form

$$e(t) = U \cdot \sin Wt)$$

(the natural frequency of the system agrees approximately with the frequency w).

The right part having the form: sin ωt

it can be obtained as the result of solving the system:

$$\frac{du1}{dt} - Wu2 = 0 \qquad \text{for } U_{10} = U$$

$$\frac{du2}{dt} + wu_1 = 0 \qquad u_{20} = 0 \tag{20}$$

Finally the system of equations assumes the following form:

$$\left.\begin{array}{l} \frac{du1}{dt} - wu_2 = 0 \\ \frac{du2}{dt} + wu_1 = 0 \\ b_{33}\frac{du3}{dt} - a_{32}u_2 + a_{34}u_4 = 0 \\ b_{44}\frac{du4}{dt} - a_{43}u_3 = 0 \end{array}\right\} \quad \begin{array}{l} \text{for } u_{10} = \frac{U}{a32} \\ U_{20} = u_{30} = u_{40} = U \\ \left\{ w = \sqrt{\frac{a_{32} \cdot a_{43}}{b_{33} \cdot b_{44}}} \right\} \quad (w = \omega) \end{array} \tag{21}$$

The solution of such a system is:

$$u_4(t) = \frac{k}{2w2} \sin Wt - \frac{k}{2W} \cdot t \cos wt \tag{22}$$

The curve in the initial part for a duration of 6 to 7 periods is in good agreement with the drawn one (Figure 11).

This is one of the cases where the error bears a special character (the case of multiple roots): if the frequency of the right part and the natural frequency of oscillations differs somewhat, then in place of a monotonically increasing oscillations we obtain the sharply differing picture of beats. In solving such a problem by an approximate method we unavoidably obtain beats instead of monotonous growth, since the accurate coincidence of the frequencies is practically impossible.

We give an example of the solution of a system of three second-order equations:

$$\frac{d^2u_1}{dt^2} + 2\lambda u_1 - \lambda u_2 = 0 \qquad \text{for } u_{10} = u_{20} = u_{30} = 0$$

$$\frac{d^2u_2}{dt^2} - \lambda u_1 + 2\lambda u_2 - \lambda u_3 = 0$$

$$\frac{d^2u_3}{dt^2} - \lambda u_2 + 2\lambda u_3 = 0 \qquad u'_{10} = u'_{30} = 0 \qquad u'_{20} = U \tag{23}$$

Introducing new variables in place of the first derivatives, we obtain the following first-order equations:

$$\frac{dx}{dt} + 2\lambda u_1 - \lambda u_2 = 0 \qquad \text{for}$$

$$\frac{dy}{dt} - \lambda u_1 + 2\lambda u_2 - \lambda u_3 = 0$$

$$\frac{dz}{dt} - \lambda u_2 + 2\lambda u_3 = 0 \tag{24}$$

$$\frac{du_1}{dt} - x = 0 \qquad u_{10} = u_{20} = u_{30} = x_0 = z_0 = 0$$

$$\frac{du_2}{dt} - y = 0 \qquad y_0 = U$$

$$\frac{du_3}{dt} - z = 0$$

The solution of this system reduces to:

Figure 12a: $u_1(t) = u_3(t)$

Figure 12b: $u_2(t)$

Figure 12c: $x(t) = z(t)$

Figure 12d: $y(t)$

All the curves are in good agreement with the drawn ones according to the analytical solution; the error in frequency and in amplitude is in the limits of 2%.

The work with the apparatus showed that the process of solving equations is very simple and graphically descriptive. It is especially simple to obtain the family of solutions for any given variation of the parameters, since to do this it is sufficient to interconnect corresponding capacitances and resistances, which is very easily effected.

The accuracy of the solution on the apparatus was of the same order as those obtained when the same calculations were carried out with the aid of a logarithmic slide rule.

At present a new and more perfect apparatus has already been made, constructed on the same principle. In this apparatus several other more complicated problems have been solved.

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(Figures attached)

ANNEX

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Figure 1

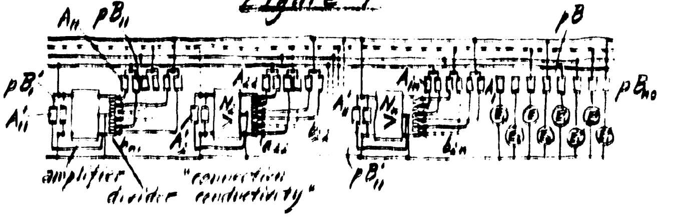


Figure 2

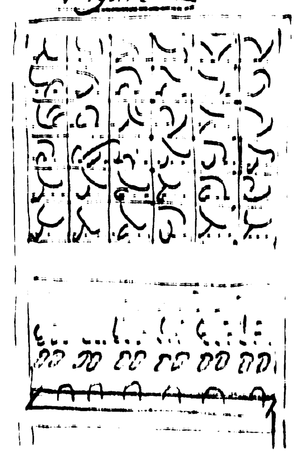
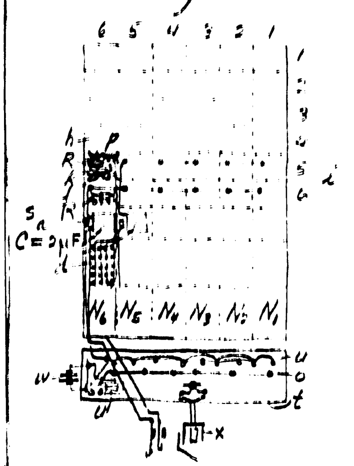


Figure 3



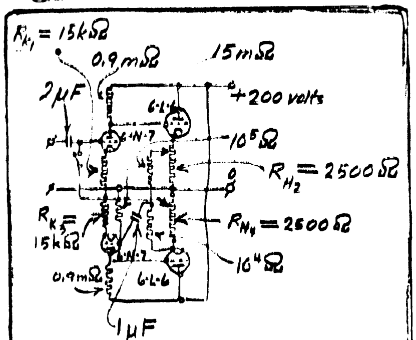


Figure 4

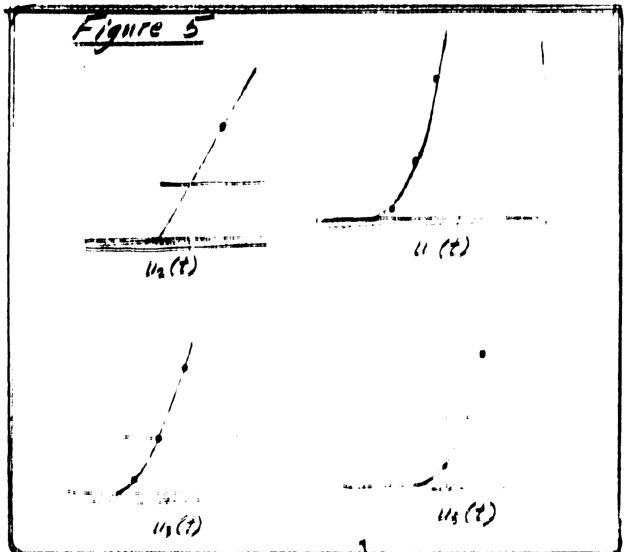


Figure 5

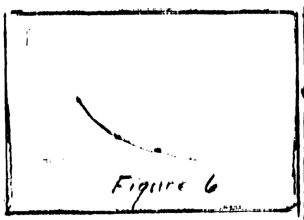


Figure 6

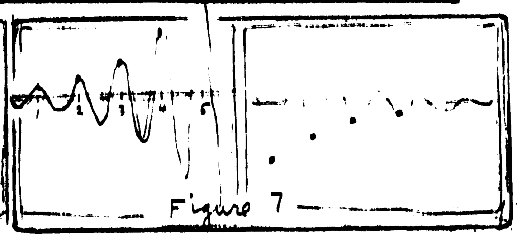


Figure 7

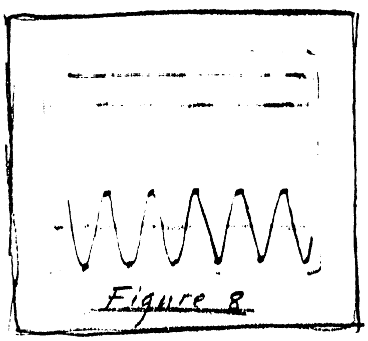


Figure 8

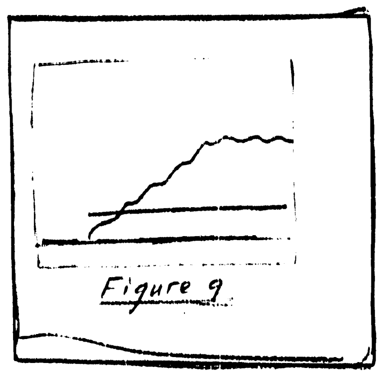


Figure 9

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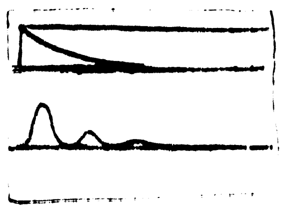

Figure 10

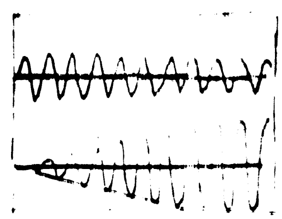

Figure 11

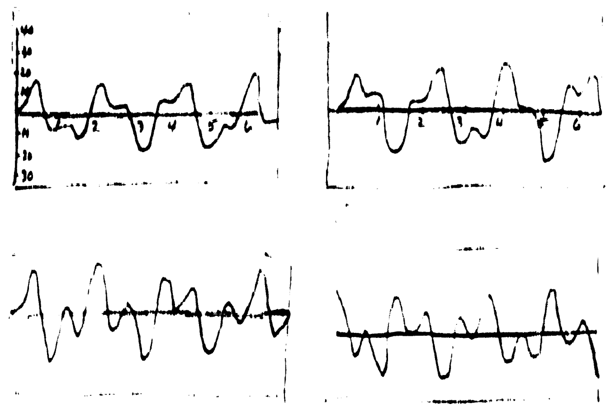

Figure 12

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